Spatial Reasoning with Incomplete Information on Relative Positioning

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Abstract

We address the problem of inferring a position in the plane from several cascaded relative positionings. A probabilistic approach is chosen, relying on known or assumed distributions of points and reference points. We derive exact formulas which are solved in typical cases of uniform or Gaussian random distributions within rectangular or circular images. This approach is illustrated with two different simulations: one is devoted to mobile phone localization, the second one to geo-positioning within a city.

Keywords

Probabilistic geometry, spatial reasoning, geometrical inference.

I. Introduction

It is an amazing capability of the human being to be able to find his/her way under very incomplete references. When visiting a city, you may find without (too much) difficulty the small souvenir shop which lies, "in the direction of the cathedral" and "on the left when walking towards the sea". Such an ability may be called "spatial reasoning". Indeed spatial reasoning consists in representing knowledge about spatial entities and spatial relationships and in reasoning on them. It is definitively very different

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from the most classical geometrical reasoning which, for instance, allows to solve, for the third side, a triangle of which two sides and an angle are known. To address the common tourist problem that was presented above, our reasoning is made of several deductions which are derived under uncertainty and incompleteness.

At first, the pieces of information are not accurate. "On the left of" and "in the direction of" are instances of information which have no exact value, and, even more, have no fixed uncertainty. They convey a rather loose meaning which may be interpreted in different ways depending on the context. These elements carry the "fuzzy" part of the spatial information which is dealt with in spatial reasoning.

Then, no piece of information is complete enough to solve the problem, and they do not share any reference which would allow to project them in a consistent framework. This constitutes the "incomplete" part of the spatial information.

In this paper, we will only address the second part of spatial reasoning, the one devoted to the "incompleteness" of information. Fuzziness of the information will not be considered here, although it is an important component of spatial reasoning.

We deal in this paper with a limited case of incomplete spatial reasoning, where we have only directional relative information between the different elements of the scene. We adopt a probabilistic description of the problem, which allows us to incorporate in a convenient way pieces of knowledge whenever they exist.

Several scientific communities addressed the problem of spatial knowledge representation. Two main classes of methods can be distinguished. The first class consists of qualitative representations and are usually based on formal logics [1], [2]. Typically spatial entities are elements of the language or propositional terms, while relationships are expressed as operators, modalities, etc. (see e.g. [1] for a survey). The second one consists of (semi-)quantitative approaches and are often based on fuzzy set or probability theories. While qualitative methods are most often applied to geographic information systems (GIS) and natural language processing, quantitative approaches are mostly found in image processing, computer vision, robotics. As far as directional relations are concerned, qualitative representations are less developed than topological relationships. Cardinal directions (i.e. North, South, East, West) are used in [3]. Other approaches are inspired by the temporal interval representations [4], and one of the most used representations (in particular in GIS) is 2D strings [5] which use relations between the projections of the considered objects on two orthogonal axes and interval-based representations on each axis. Let us finally mention the approach in [6] which represents the relative position of a point with respect to two other ones as a 5×3 matrix based on a subdivision of the space into six sectors related to the two reference points. Quantitative representations of directional relations are perhaps more developed. The ambiguity of such relations led to fuzzy representations, already suggested in [7]. Most of existing methods for defining fuzzy relative spatial position rely on angle measurements between points of the two objects of interest [8], [9], and concern 2D objects. A fuzzy relationship is defined as a fuzzy set, and the correspondence between the relation and the angle measurements is then evaluated. This approach has been extended in order to represent both distances and angles in a bi-dimensional histogram in [10]. A method based on linear cross-sections of the objects instead of only points has been developed in [11]. Finally, methods based on whole objects have been proposed, based either on learning from human evaluation [12], on projections of the objects [13], or on a morphological approach [14]. A detailed comparison of these approaches can be found in [15].

In this paper, we propose a quantitative representation based on probabilities.

While the problem of inference and reasoning about spatial relations [16] has been widely addressed in logical and qualitative representations, where the strong apparatus of formal logics is very useful, it is still in its infancy when quantitative representations are used, and much less developments have been done. This paper is a contribution towards this aim. More specifically we will model the inference of the relations between two points from knowledge about the relations of each of these points to a third one.

This problem is precisely stated and formalized in Section II. In particular we detail the cases where we assume that points are uniformly distributed in a bounded space (circle or rectangle) in Section III, and in the case of a Gaussian distribution in Section IV. Inference is then performed in a Bayesian framework (Section V). Two illustrative examples are described in Section VI, one dealing with small objects such as in the case of localization of mobile phones, and one addressing the problem of georeferencing in a city.

II. Problem statement and probabilistic formulation

We consider the problem of localization where points are only known through the direction where they are with respect to a reference point, and not through their distance to this point. More precisely, we address the simplest problem which may be stated as[1]:

Problem 1: *Let C be an unknown point. Let C be in the direction β from a point B, this point B being itself in the direction α with respect to a reference point A. What can we say about the position of point C with respect to A?*

We will demonstrate that, under rather loose assumptions, if we know the statistical distribution of the unknown points, we may derive the exact distribution function of point C and propose good estimates of its relative positioning with respect to A.

We first introduce some notations. We choose, without loss of generality, point A as the origin of the Cartesian plane limited to a domain $\mathcal{F}$ (called the image in the following). Any point M may be represented by its Cartesian coordinates x_M or y_M, or by its polar coordinates r_M and θ_m. As a convention, r_M is a positive or null number and θ_M belongs to $[-\pi, \pi[$. The notation $[x \pm dx]$ denotes

[1] Directions in the 2D space are defined by the angle with respect to the horizontal axis.

the interval $[x - dx, x + dx]$.

We consider that points M are distributed in the image according to a random distribution $f_M(r_M, \theta_M)$:

$$f_M(r_M, \theta_M) = \lim_{drd\theta \to 0} P(M : r_M \in [r \pm dr/2], \theta_M \in [\theta \pm d\theta/2]) \; / \; rdrd\theta.$$

We will need in the sequel that f_M be a continuous function of the two variables r_M and θ_M.

Figure 1 represents the configuration expressed in Problem 1.

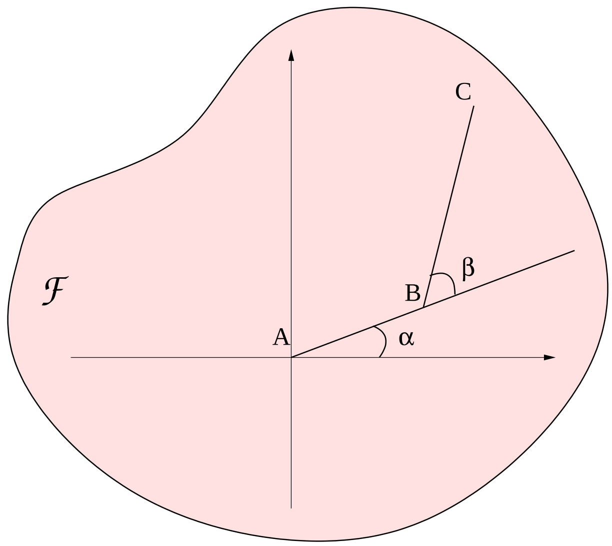


Fig. 1. Problem 1 may be stated as: "Where is point C with respect to A knowing that it is in direction β with respect to B, itself in the direction α with respect to A?". The domain of research is limited to $\mathcal{F}$.

We are interested in the probability $P(C, \alpha, \beta)$ which expresses the probability distribution of point C and the two angles α and β. According to Figure 2, this probability is equal to:

$$P(C, \alpha, \beta) = P(C) \int_{\Sigma} P(B) dB, \tag{1}$$

where $\Sigma = S_{A,\alpha} \cap S_{C,\pi-\beta}$, and $S_{A,\alpha}$ and $S_{C,\pi-\beta}$ are the two angular sectors described in Figure 2.

The domain Σ is limited by four lines with equations:

$$r = r_c \frac{\sin[\theta_c - (\beta - \pi \pm d\beta/2)]}{\sin[\theta - (\beta - \pi \pm d\beta/2)]},$$

$$\theta = \alpha \pm d\alpha/2.$$

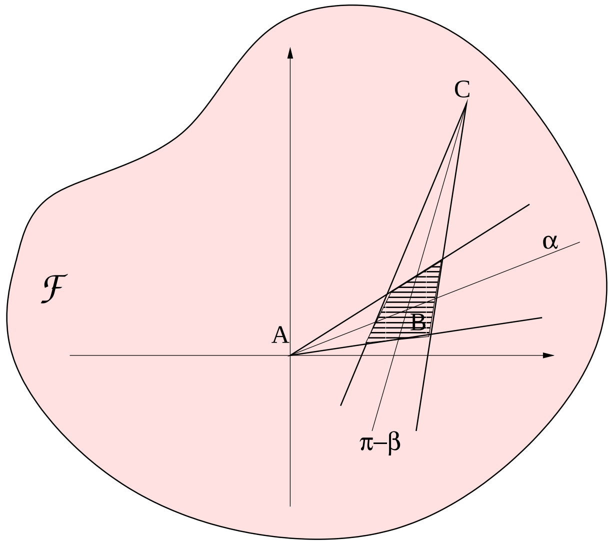


Fig. 2. Every point B in the hashed area fulfills the two conditions to be in the direction α with respect to point A (sector $S_{A,\alpha}$) and to have point C in the direction β (sector $S_{C,\pi-\beta}$) within the respective tolerances $\pm\, d\alpha/2, \pm\, d\beta/2$.

Denoting by $g_{r,\theta}(\phi)$ the function of $\mathbb{R} \to \mathbb{R}$:

$$g_{r,\theta}(\phi) = r\frac{\sin[\theta_c - (\phi - \pi)]}{\sin[\theta - (\phi - \pi)]},$$

Equation 1 becomes:

$$P(C,\alpha,\beta) = f_C(r_c,\theta_c) \lim_{d\alpha d\beta \to 0} \frac{1}{d\alpha d\beta} \int_{\alpha - d\alpha/2}^{\alpha + d\alpha/2} \int_{g_{r_c,\theta_b}(\beta - d\beta/2)}^{g_{r_c,\theta_b}(\beta + d\beta/2)} f_B(r_b,\theta_b) r_b dr_b d\theta_b.$$

If the function f_B is continuous, some computation leads to:

$$P(C,\alpha,\beta) = 2r_c^2 \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta-2\theta_c)}{3\sin(\alpha-\beta) - \sin(3\alpha - 3\beta)} f_C(r_c,\theta_c) f_B(r_c \frac{\sin(\theta_c - \beta)}{\sin(\alpha-\beta)}, \alpha). \tag{2}$$

This probability distribution (Equation 2) holds for any continuous distribution of points B in the plane.

III. Solution in the case of a uniform distribution

We consider here the case of a uniform distribution. We first express $P(C,\alpha,\beta)$ for this distribution, from which we derive $P(\alpha,\beta)$. Finally, we compute $P(C|\alpha,\beta)$ and $P(\gamma|\alpha,\beta)$ from which inference will be performed (see Section V).

A. In a circular image

Let us now assume that the points B and C have a uniform distribution in a circular image of radius R and origin A, i.e.:

$$f_C(r,\theta) = f_B(r,\theta) = f(r,\theta) = \frac{1}{\pi R^2} = K.$$

Then Equation 2 becomes:

$$P(C,\alpha,\beta) = 2r_c^2 K^2 \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta-2\theta_c)}{3\sin(\alpha-\beta) - \sin(3\alpha-3\beta)}. \quad (3)$$

Figure 3 illustrates this distribution for $\alpha = \frac{\pi}{6}$ and $\beta = \frac{2\pi}{3}$, and Figure 4 for $\alpha = \frac{\pi}{6}$ and $\beta = \pi$.

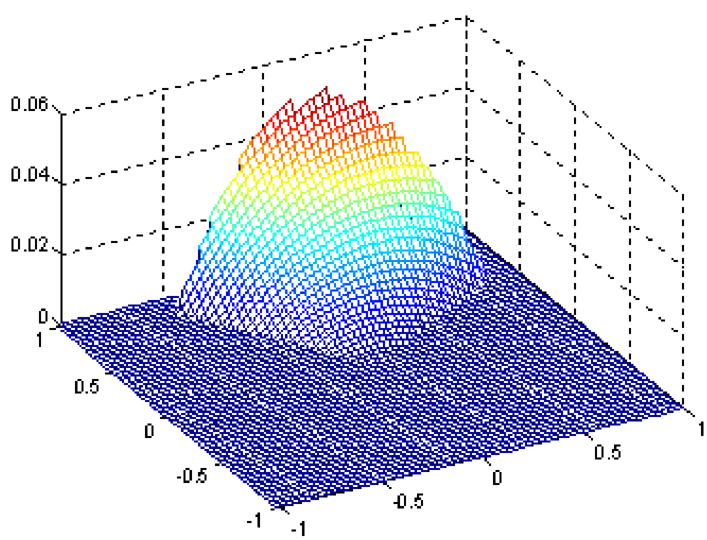


Fig. 3. $P(C,\alpha,\beta)$ for $\alpha = \frac{\pi}{6}$ and $\beta = \frac{2\pi}{3}$.

The probability $P(C,\alpha,\beta)$ has to be computed in an angular sector of radius R, thus limiting its support to a region $D_{\alpha,\beta}$. This region corresponds to the area covered by the angular sector $S_{B,\beta}$ of origin B, angle β and aperture $d\beta$ when B belongs to the angular sector $S_{A,\alpha}$ (of origin A, angle α and aperture $d\alpha$). This region can have two different shapes depending on $\beta - \alpha$ (see Figure 5):

- if $(\beta-\alpha) \in]2k\pi, \pi/2 + 2k\pi]$, then $D_{\alpha,\beta}$ is a complete angular sector;

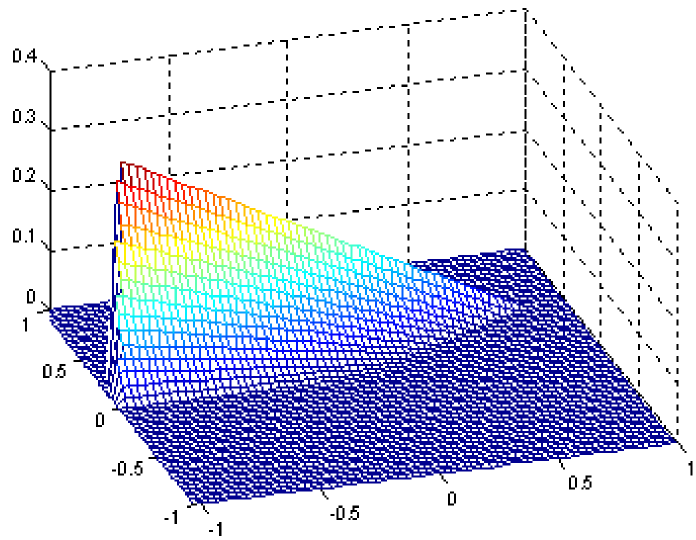


Fig. 4. $P(C, \alpha, \beta)$ for $\alpha = \frac{\pi}{6}$ and $\beta = \pi$.

- if $(\beta - \alpha) \in [\pi/2 + 2k\pi, (2k+1)\pi[$, then $D_{\alpha,\beta}$ consists of a complete angular sector (for $\theta \in [2\beta - \alpha - \pi, \beta]$) and a triangle (for $\theta \in [\alpha, 2\beta - \alpha - \pi]$).

This is illustrated in Figure 5.

We can now derive an expression for $P(\alpha, \beta)$:

$$P(\alpha, \beta) = \int_{D_{\alpha,\beta}} P(C, \alpha, \beta) dC,$$

which leads to two different expressions depending on the shape of $D_{\alpha,\beta}$. In the first case (complete angular sector), we get:

$$P(\alpha, \beta) = \frac{(\alpha - \beta)\cos(\alpha - \beta) - \sin(\alpha - \beta)}{2\pi^2(\sin(3\alpha - 3\beta) - 3\sin(\alpha - \beta))}, \tag{4}$$

while in the second case (complete angular sector plus triangle), we get:

$$P(\alpha, \beta) = \frac{5\sin(\alpha - \beta) + \sin(3\alpha - 3\beta) - 3\sin(5\alpha - 5\beta) + \sin(7\alpha - 7\beta)}{8\pi^2(10\sin(\alpha - \beta) - 5\sin(3\alpha - 3\beta) + \sin(5\alpha - 5\beta))}$$
$$+ \frac{2(\beta - \alpha - \pi)\cos(\alpha - \beta) - \sin(\alpha - \beta) + \sin(3\alpha - 3\beta)}{4\pi^2(\sin(3\alpha - 3\beta) - 3\sin(\alpha - \beta))}. \tag{5}$$

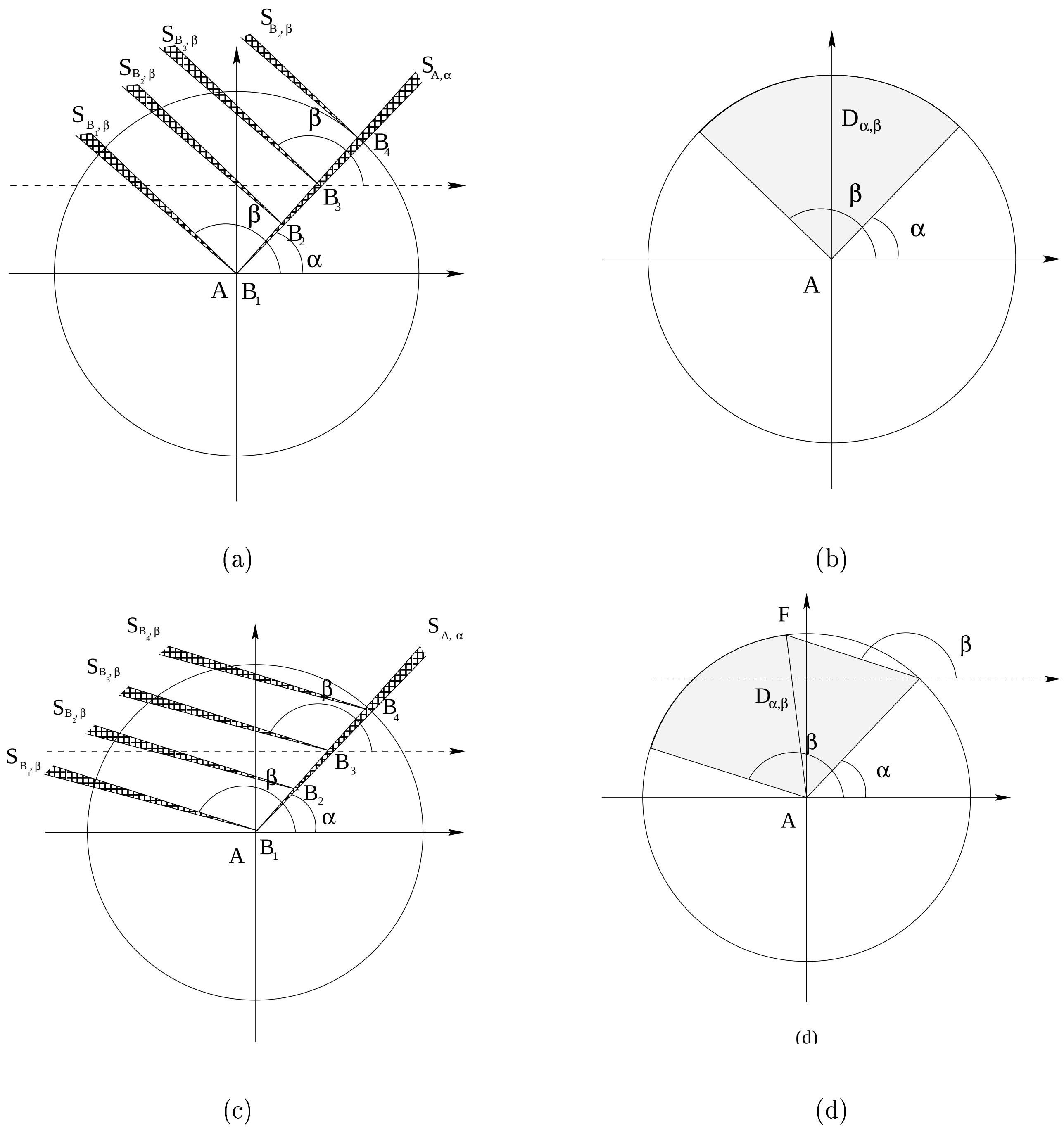


Fig. 5. Definition of the domain $D_{\alpha,\beta}$ and its two possible shapes.

By setting

$$K_1 = \frac{2K^2}{P(\alpha,\beta)},$$

which takes two different expressions depending on the value of $\beta - \alpha$, the conditional probability $P(C/\alpha,\beta)$ is expressed as:

$$P(C/\alpha,\beta) = \begin{cases} K_1 r_c^2 \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\theta_c)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} & \text{if } C \in D_{\alpha,\beta}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Now if we want to assess the spatial relations between C and A, we have to compute the probability of angle γ (between A and C) conditionally to α and β:

$$P(\gamma/\alpha,\beta) = \int_{S_{A,\gamma}\cap D_{\alpha,\beta}} P(C/\alpha,\beta)dC, \tag{7}$$

where $S_{A,\gamma}$ is the angular sector of origin A, angle γ and aperture $d\gamma$. The probability $P(\gamma \,/\, C,\alpha,\beta)$ is equal to 1 if C is in this sector and to 0 if it is outside, which explains the above formula.

Again we have to distinguish between two cases:

- if $(\beta-\alpha) \in]2k\pi, \pi/2+2k\pi]$, we obtain:

$$P(\gamma/\alpha,\beta) = \begin{cases} K_1 \frac{R^4}{4} \frac{cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} & \text{if } \alpha \le \gamma \le \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

- if $(\beta-\alpha) \in [\pi/2+2k\pi, (2k+1)\pi[$, we obtain:

$$P(\gamma/\alpha,\beta) = \begin{cases} K_1 \frac{(R\frac{\sin(\beta-\alpha)}{\sin(\beta-\gamma)})^4}{4} \frac{cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} & \text{if } \alpha \le \gamma \le 2\beta-\alpha-\pi, \\ K_1 \frac{R^4}{4} \frac{cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} & \text{if } 2\beta-\alpha-\pi \le \gamma \le \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Figure 6 illustrates the shape of $P(\gamma/\alpha,\beta)$ for $\alpha = \frac{\pi}{6}$ and for $\beta = \frac{\pi}{2}$, $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$ respectively. The theoretical curves are compared with numerical simulations obtained from random positions for B and C generated according to a uniform distribution. Very similar shapes are obtained.

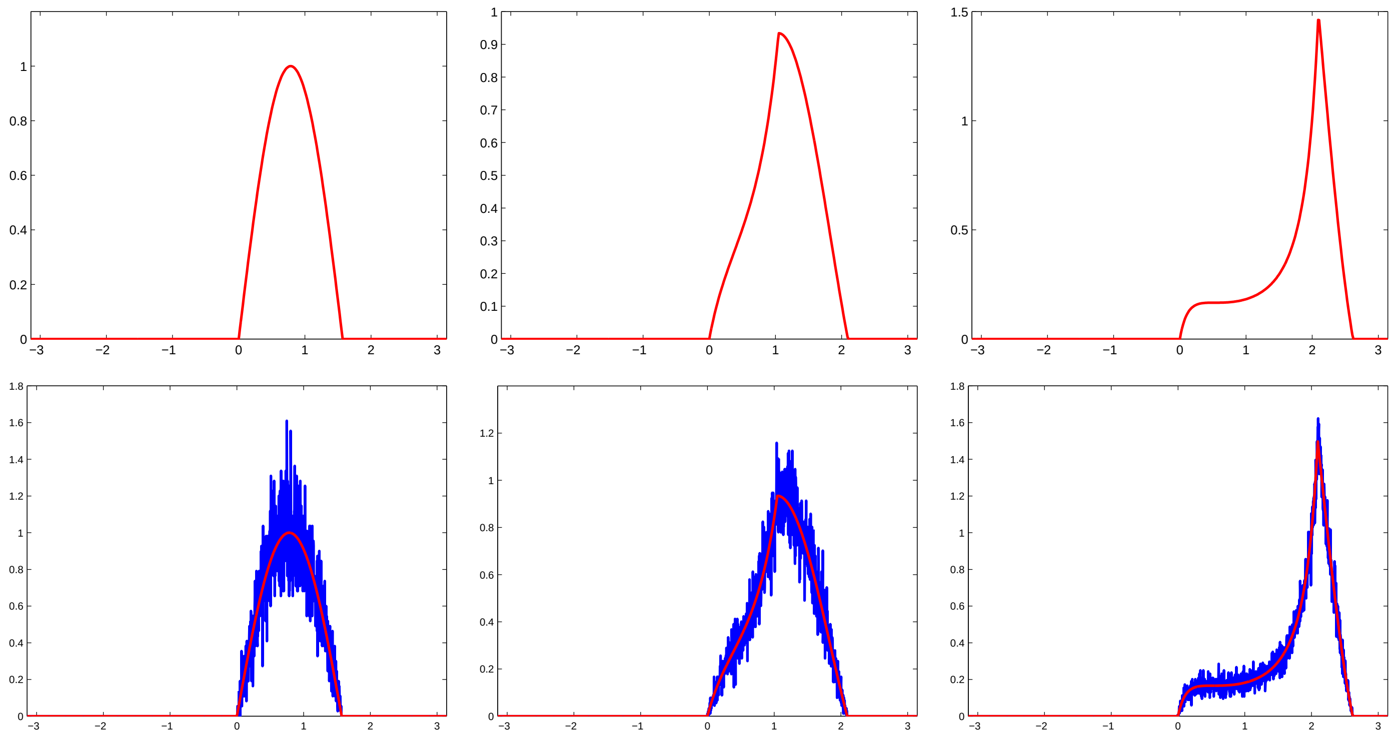


Fig. 6. $P(\gamma/\alpha, \beta)$ for $\alpha = 0$ and for $\beta = \frac{\pi}{2}$, $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$. Red line: theoretical curves. Blue lines: numerical simulations.

In the limit cases where $\beta = \alpha + k\pi$, different equations are obtained:

- for even values of k, we have:

$$P(C/\alpha, \beta) = \begin{cases} \frac{4r_c^2}{R^4} & \text{if } C \in S_{A,\alpha}, \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

$$P(\gamma/\alpha, \beta) = \begin{cases} 1 & \text{if } \gamma = \alpha + k\pi, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

- for odd values of k, we have:

$$P(C/\alpha, \beta) = \begin{cases} \frac{12(R-r_c)^2}{7R^4} & \text{if } C \in S_{A,\alpha}, \\ \frac{12}{7R^2} & \text{if } C \in S_{A,\alpha+\pi}, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

$$P(\gamma/\alpha,\beta) = \begin{cases} \frac{1}{7} & \text{if } \gamma = \alpha + (k-1)\pi, \\ \frac{6}{7} & \text{if } \gamma = \alpha + k\pi, \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

B. In a square image

Let us now consider the case of a square image of side length equal to $2R$, still with a uniform distribution. The formula for $P(C,\alpha,\beta)$ remains the same, except for the constant K which is now:

$$K = \frac{1}{4R^2},$$

R being the half length of the side of the square. In this Section, we just give the obtained results for $P(\alpha,\beta)$ and $P(\gamma|\alpha,\beta)$. The detailed computation can be found in [10].

The domain $D_{\alpha,\beta}$ is more difficult to compute than for the circular window, and more cases have to be distinguished. To reduce the number of cases, we assume that $\alpha \in [-\pi/4, \pi/4]$ (the other cases are easily deduced from this one) and $\beta \in]\alpha, \alpha+2\pi[$ (without loss of generality).

Figure 7 represents the possible shapes of $D_{\alpha,\beta}$. The last five cases are symmetrical to the first five ones, therefore only the first five cases are dealt with in the following.

1. Case $\alpha < \beta \leq \pi/4$:

$$P(\gamma/\alpha,\beta) = \begin{cases} \frac{1}{32P(\alpha,\beta)} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))\cos^4\gamma} & \text{if } \alpha \leq \gamma \leq \beta, \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

with

$$P(\alpha,\beta) = \frac{1}{384\cos^2\alpha\cos^2\beta}.$$

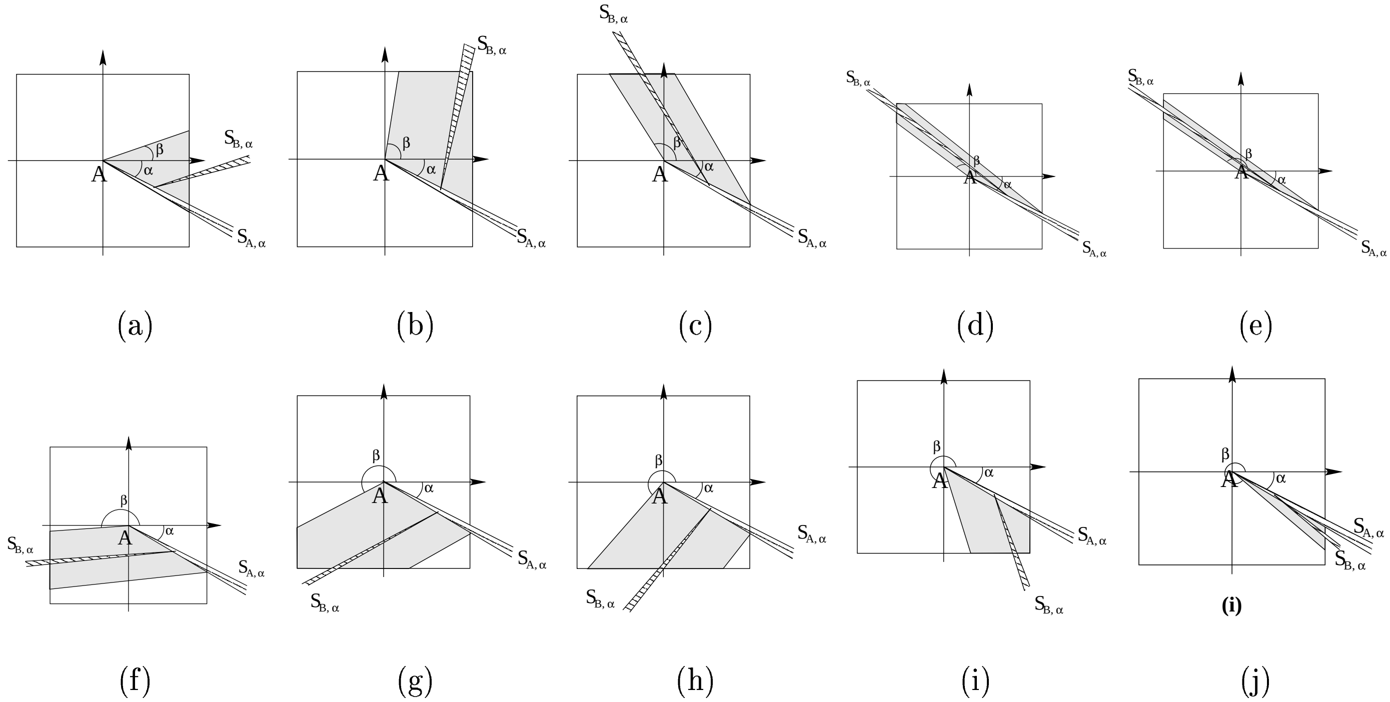


Fig. 7. Possible shapes of $D_{\alpha,\beta}$ for a square window.

2. Case $\pi/4 \le \beta \le \pi/2$:

$$P(\gamma/\alpha,\beta) = \begin{cases} \frac{1}{32P(\alpha,\beta)} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))\cos^4\gamma} & \text{if } \alpha \le \gamma \le \pi/4, \\ \frac{1}{32P(\alpha,\beta)} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))\sin^4\gamma} & \text{if } \pi/4 \le \gamma \le \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

with

$$P(\alpha,\beta) = \frac{\cos^2(\alpha+\pi/4)(8\sin(\alpha-\beta)-4\sin(\alpha+\beta)+4\cos(\alpha+\beta)+4\cos(\alpha-\beta))}{192\cos^2\alpha(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))}$$
$$+\frac{\cos(\beta+\pi/4)^2(8\sin(\alpha-\beta)+4\cos(\alpha-\beta)-4\cos(\alpha+\beta)-4\sin(\alpha+\beta))}{192\sin^2\beta(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))}.$$

3. Case $\pi/2 \le \beta \le 3\pi/4$:

$$P(\gamma/\alpha,\beta) = \begin{cases} \frac{1}{32P(\alpha,\beta)} \frac{(\cos(\beta-\alpha)-\cos(\alpha+\beta-2\gamma))\sin^4(\alpha-\beta)}{\sin^4(\gamma-\beta)\cos^4\alpha(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))} & \text{if } \alpha \le \gamma \le \varphi, \\ \frac{1}{32P(\alpha,\beta)} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))\sin^4\gamma} & \text{if } \varphi \le \gamma \le \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

with

$$P(\alpha,\beta)=\frac{\sin^2(\varphi-\alpha)\sin^2(\varphi+\beta)}{128\cos^4\alpha(\cos^2\beta-\cos^2\varphi)^2}$$
$$+\frac{\sin^2(\beta-\varphi)(3\cos(\alpha-\beta-\varphi)-\cos(\alpha-\beta+\varphi)-2\cos(\alpha+\beta-\varphi))}{192\sin^3\varphi\sin^3\beta(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))},$$

and φ is the angle limiting the two triangles building the domain $D_{\alpha,\beta}$ (see Figure 8):

$$\varphi=\operatorname{arccotg}\frac{\cos\alpha\cos\beta+\sin(\beta-\alpha)}{\cos\alpha\sin\beta}.$$

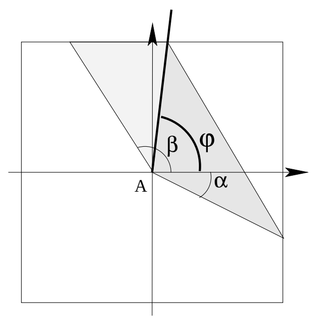


Fig. 8. Support of probability $P(C,\alpha,\beta)$ in the case $\pi/2\le\beta\le3\pi/4$.

4. Case $\beta\ge3\pi/4$ and $\varphi\le3\pi/4$:

$$P(\gamma/\alpha,\beta)=\begin{cases}\frac{1}{32P(\alpha,\beta)}\frac{(\cos(\beta-\alpha)-\cos(\alpha+\beta-2\gamma))\sin^4(\beta-\alpha)}{\sin^4(\gamma-\beta)\cos^4\alpha(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))} & \text{if } \alpha\le\gamma\le\varphi,\\ \frac{1}{32P(\alpha,\beta)}\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))\sin^4\gamma} & \text{if } \varphi\le\gamma\le3\pi/4,\\ \frac{1}{32P(\alpha,\beta)}\frac{\cos(\alpha+\beta-2\gamma)-\cos(\beta-\alpha)}{(3\sin(\beta-\alpha)+\sin(3\alpha-3\beta))\cos^4\gamma} & \text{if } 3\pi/4\le\gamma\le\beta,\\ 0 & \text{otherwise,}\end{cases}\tag{17}$$

with

$$P(\alpha,\beta)=\frac{\sin^2(\varphi-\alpha)\sin^2(\varphi+\beta)}{128\cos^4\alpha(\cos^2\beta-\cos^2\varphi)^2}+\frac{4\cos(\alpha-\beta)+2\sin(\alpha+\beta)-\sin(\alpha-\beta)}{48(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))}$$
$$-\frac{\cos\varphi\cos(\alpha-\beta)(2\cos^2\varphi-3)+\cos(\alpha+\beta-3\varphi)}{96\sin^3\varphi(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))}$$
$$+\frac{\sin(\alpha+\beta)+2\sin(\alpha-\beta)}{96\cos^2\beta(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))}.$$

5. Case $\beta < \alpha + \pi$ and $\varphi \geq 3\pi/4$:

$$P(\gamma/\alpha,\beta) = \begin{cases} \frac{1}{32P(\alpha,\beta)} \frac{(\cos(\beta-\alpha)-\cos(\alpha+\beta-2\gamma))\sin^4(\beta-\alpha)}{\sin^4(\gamma-\beta)\cos^4\alpha(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))} & \text{if } \alpha \leq \gamma \leq \psi, \\ \frac{1}{32P(\alpha,\beta)} \frac{\cos(\beta-\alpha)-\cos(\alpha+\beta-2\gamma)}{(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))\cos^4\gamma} & \text{if } \psi \leq \gamma \leq \beta, \\ 0 & \text{otherwise,} \end{cases} \quad (18)$$

with

$$P(\alpha,\beta) = \frac{\sin^2(\psi-\alpha)\sin^2(\psi+\beta)}{128\cos^4\alpha(\cos^2\beta-\cos^2\psi)^2}$$
$$+\frac{\sin^2(\beta-\psi)(2\sin(\beta-\psi+\alpha)-3\sin(\beta+\psi-\alpha)-\sin(\beta-\psi-\alpha))}{192\cos^3\psi\cos^2\beta(3\sin(\alpha-\beta)-\sin(3\alpha-3\beta))},$$

and

$$\psi = \text{arccotg}\left(\frac{\cos(\beta-\alpha)+\cos(\beta+\alpha)}{\sin(\beta+\alpha)+3\sin(\beta-\alpha)}\right);$$

ψ is the angle limiting the two triangles building the support $D_{\alpha,\beta}$ of the probability $P(C,\alpha,\beta)$ (see Figure 9).

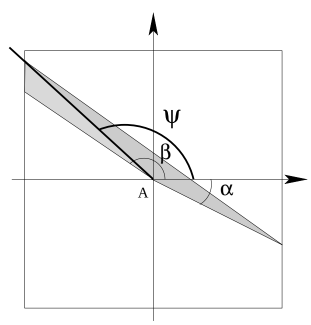


Fig. 9. Support of the probability $P(C,\alpha,\beta)$ in the case $\beta < \alpha + \pi$ and $\varphi \geq 3\pi/4$.

IV. Solution in the case of a Gaussian distribution

Since Equation 2 holds for any distribution, the previous computations can be extended to other distributions, such as Gaussian distributions for instance [10]. Let us briefly present the results obtained in this case, for a circular window.

We assume that the point distribution has the following form:

$$f(r,\theta) = \frac{1}{2\pi} e^{-\frac{r^2}{2}}. \tag{19}$$

In the general case where $\beta \neq \alpha + k\pi$ $(k \in \mathbb{Z})$, we obtain:

$$P(C,\alpha,\beta) = \frac{r_c^2(\cos(\alpha-\beta) - \cos(\alpha+\beta-2\theta_c))}{2\pi^2(3\sin(\alpha-\beta) - \sin(3\alpha-3\beta))} e^{-\frac{r_c^2}{2}(1+(\frac{\sin(\theta_c-\beta)}{\sin(\alpha-\beta)})^2)}. \tag{20}$$

The derivation of probability $P(\gamma,\alpha,\beta)$ leads to:

$$P(\gamma,\alpha,\beta) = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta-2\gamma)}{\pi^2(3\sin(\alpha-\beta) - \sin(3\alpha-3\beta))(\frac{\sin^2(\beta-\gamma)}{\sin^2(\beta-\alpha)}+1)^2}. \tag{21}$$

As for the other distributions, the special case of $\beta = \alpha + k\pi$ $(k \in \mathbb{Z})$ needs specific consideration and different equations are obtained, that are not detailed here (see [10] for more results).

Figure 10 illustrates the obtained results for various values of α and β.

V. Bayesian decision

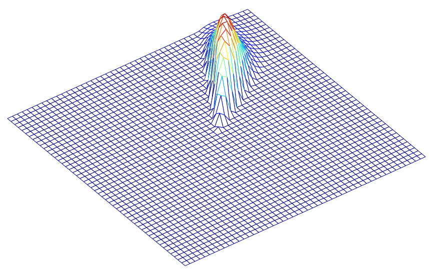
We now address the problem of inference, defined here as a decision rule $\mathcal{D}(\alpha,\beta)$ which assigns to observations α and β an angle γ corresponding to the directional relative position of point C with respect to point A. This decision rule has to minimize a cost function that can be defined in different ways. Let us denote by $\gamma^*(\alpha,\beta)$ the exact position of point C with respect to point A.

For instance, the cost function could be the mean decision error over the complete set of distributions of points. The global error probability is expressed as:

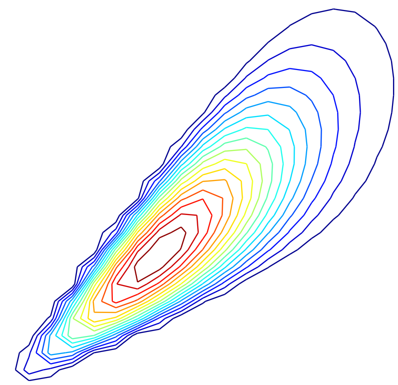
$$P_{err} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\mathcal{D}(\alpha,\beta) \neq \gamma^*(\alpha,\beta))P(\alpha,\beta)d\alpha d\beta. \tag{22}$$

The probability $P(\mathcal{D}(\alpha,\beta) = \gamma^*(\alpha,\beta))$ is the probability that C be in direction $\mathcal{D}(\alpha,\beta)$ conditionally to α and β. Therefore we have:

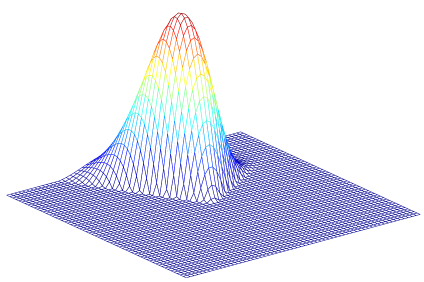
$$P_{err} = 1 - \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\mathcal{D}(\alpha,\beta)/\alpha,\beta)P(\alpha,\beta)d\alpha d\beta. \tag{23}$$



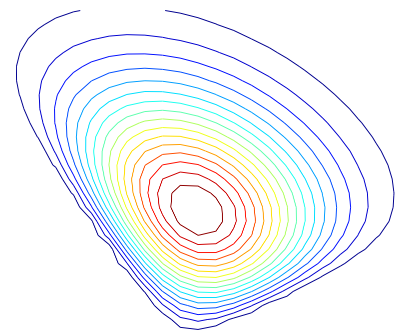
$P(C, \alpha = \pi/6, \beta = \pi/3)$.



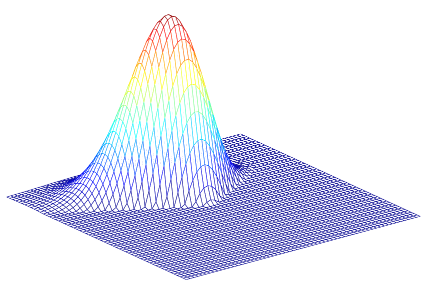
$P(C, \alpha = \pi/6, \beta = \pi/3)$.



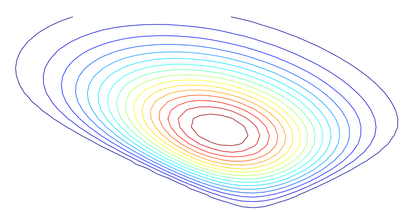
$P(C, \alpha = \pi/6, \beta = 2\pi/3)$.



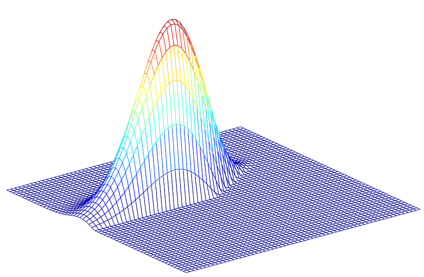
$P(C, \alpha = \pi/6, \beta = 2\pi/3)$.



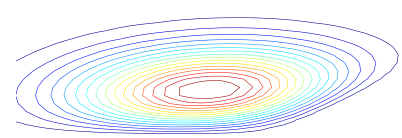
$P(C, \alpha = \pi/6, \beta = 5\pi/6)$.



$P(C, \alpha = \pi/6, \beta = 5\pi/6)$.



$P(C, \alpha = \pi/6, \beta = \pi)$.



$P(C, \alpha = \pi/6, \beta = \pi)$.

Minimizing the global error amounts to maximizing $P(\gamma/\alpha, \beta)$.

In the case of a uniform distribution in a circular window of radius R, the unique maximum of $P(\gamma/\alpha, \beta)$ is reached for two different values of γ depending on $\beta - \alpha$, which leads to:

$$\mathcal{D}(\alpha, \beta) = \gamma = \begin{cases} \frac{\alpha+\beta}{2} & \text{if } -\frac{2\pi}{3} \leq \beta - \alpha \leq \frac{2\pi}{3}, \\ 2\beta - \alpha\pi & \text{otherwise.} \end{cases} \tag{24}$$

Another possible cost function is the likelihood. Maximizing the likelihood $P(\alpha, \beta/\gamma)$ amounts to maximize the quantity:

$$\frac{P(\gamma/\alpha, \beta)}{P(\gamma)},$$

and finally to maximize the posterior probability $P(\gamma/\alpha, \beta)$ since no direction is favored. The results are therefore the same as in the previous case.

As a last example, let us consider the minimization of the average decision risk, which generalizes the first criterion by introducing a cost function $\lambda(\mathcal{D}(\alpha, \beta)/\gamma)$. Minimizing the average risk amounts to minimize the conditional risk expressed as:

$$R(\mathcal{D}(\alpha, \beta)/\alpha, \beta) = \int_{-\pi}^{\pi} \lambda(\mathcal{D}(\alpha, \beta)/\gamma) P(\gamma/\alpha, \beta) d\gamma. \tag{25}$$

For instance if λ is defined as the squared difference $(\mathcal{D}(\alpha, \beta) - \gamma)^2$, then the minimum is obtained for the following value of γ (for a uniform distribution in a disk):

$$\gamma_0 = \begin{cases} \frac{\alpha+\beta}{2} & \text{if } (\beta - \alpha) \in [2k\pi, \pi/2 + 2k\pi], \\ \frac{\psi(\alpha,\beta)}{\varphi(\alpha,\beta)} & \text{if } (\beta - \alpha) \in [2k\pi + \pi/2, (2k+1)\pi[, \\ \alpha + \frac{6\pi}{7} & \text{if } \beta = \alpha + \pi, \end{cases} \tag{26}$$

with

$$\psi(\alpha, \beta) = -4((2\beta - \alpha - \pi)^2 - \beta^2 + 1)\cos(\alpha - \beta) + 5\cos(3\alpha - 3\beta) - \cos(5\alpha - 5\beta)$$

$$-(12\alpha - 20\beta + 8\pi)\sin(\alpha - \beta) + (8\alpha - 10\beta + 5\pi)\sin(3\alpha - 3\beta) - (2\beta - \pi)\sin(5\alpha - 5\beta),$$

and

$$\varphi(\alpha,\beta) = 8(\alpha-\beta+\pi)\cos(\alpha-\beta) + 8\sin(\alpha-\beta) - 2\sin(3\alpha-3\beta) - 2\sin(5\alpha-\beta).$$

VI. Illustrative examples

A. Localization of mobile phone simulation

Let us first consider the problem of inference of directional spatial relations in case of small (quasi-points) objects, for which the previous results can be directly used. An example is given in Figure 11. The problem in this synthetic example is to find the position of point C with respect to point A, knowing its position with respect to some reference points B_i (this is a quite common situation when trying to localize mobile phones). We assume here a uniform distribution in a circular window.

In this synthetic example, the solution is known: $\theta_{AC} = 0.64 rad$. The values for γ_0 (obtained with inference based on minimization of error probability), $|\gamma_0 - \theta_{AC}|$ (estimation error with respect to the true value) and $|\beta-\alpha|$ are given in Table I for the six instances B_i of B.

Reference point	γ_0	$\|\gamma_0 - \theta_{AC}\|$	$\|\beta-\alpha\|$
B_1	0.52	0.12	2.12
B_2	0.59	0.05	1.01
B_3	0.68	0.04	1.07
B_4	0.79	0.15	2.16
B_5	0.67	0.03	2.83
B_6	0.62	0.02	2.94

TABLE I

Values of γ_0, $|\gamma_0 - \theta_{AC}|$ and $|\beta-\alpha|$ for the points of Figure 11. For low (close to 0) or large (close to π values of $|\beta-\alpha|$, the error $|\gamma_0 - \theta_{AC}|$ is small.

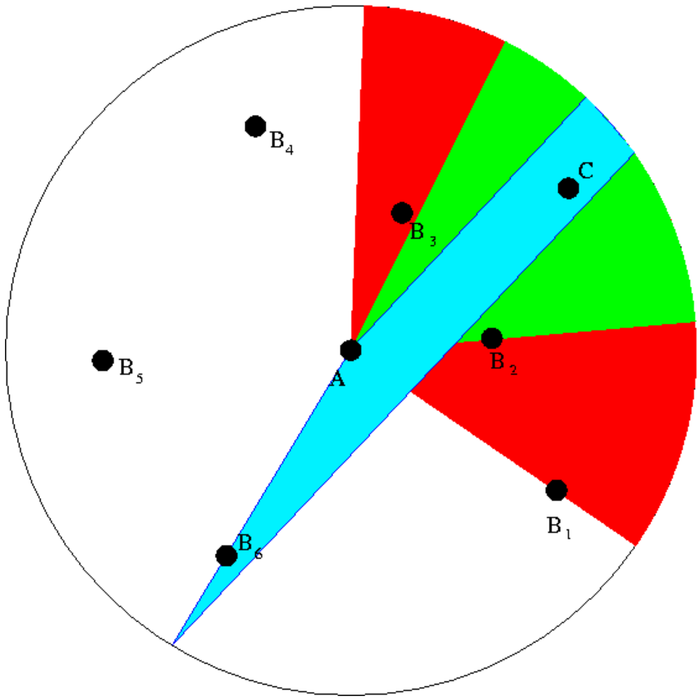


Fig. 11. Example of small objects. The reference point A is the center of the figure. One instance of point C and six instances of the intermediary point B are given ($B_1 ... B_6$).Red: angular sector for B_1, green: angular sector for B_2, blue: angular sector for B_6.

The smallest estimation errors occur for low or large values of $|\beta - \alpha|$. This was expected since the localization is more difficult if the angular sector $D_{\alpha,\beta}$ is larger.

The error probability decreases if several intermediary objects are used, by combining the corresponding probabilities $P(\gamma/\alpha, \beta)$. The probabilistic distributions are given in Figure 12, each curve corresponding to one of the B_i. All curves have a peak very close to the exact value of θ_{AC} (i.e. 0.64). They are more or less spread depending on the uncertainty that remains in the location (i.e. depending on $|\beta - \alpha|$). For instance, the angular sector for B_6 is reduced (see Figure 11), which leads to a good estimation (blue curve in Figure 12). The product of these probabilities clearly tends towards a Dirac function (corresponding to an intersection of several domains $D_{\alpha,\beta}$), showing that using several reference points improves the localization, as was expected.

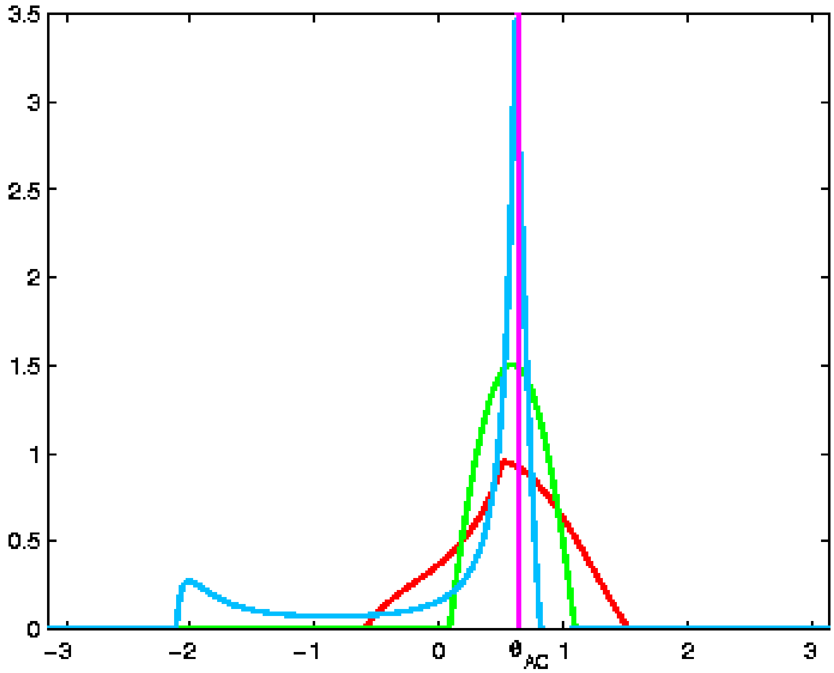


Fig. 12. Probability distributions obtained for some points of Figure 11. Red: B_1, green: B_2, blue: B_6. The vertical line indicates the correct solution.

B. Georeferencing in a city

In this example, we address a more complex situation, where we have to localize a point, here the ENST (Ecole Nationale Supérieure des Télécommunications), in a real city map, based on knowledge about relations to other geographical locations.

The map[2] of the considered area is shown in Figure 13.

If no knowledge is available, we only assume a uniform distribution of points in the plane. This

[2]http://www.maporama.com

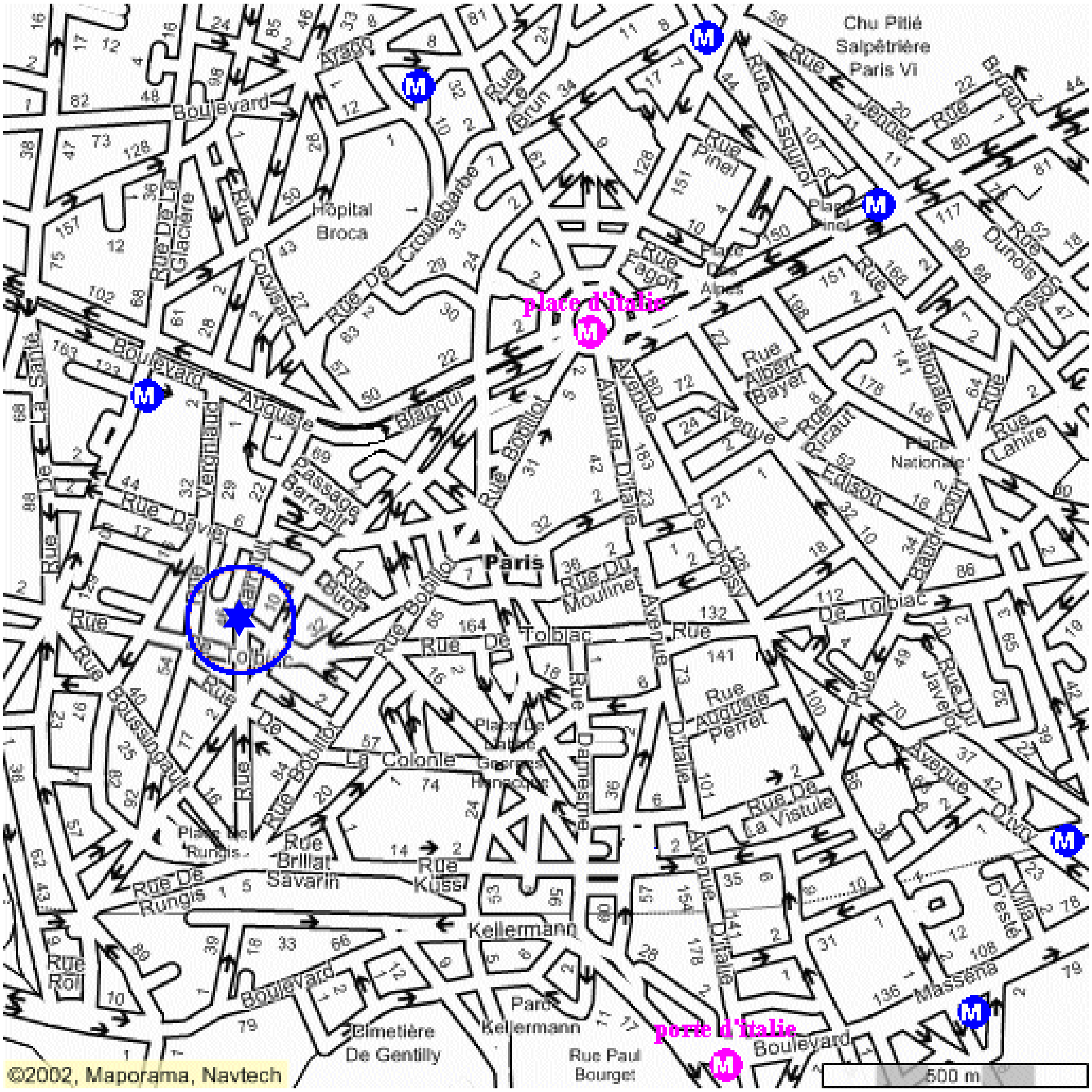


Fig. 13. Map of an area around ENST in Paris (blue circle).

distribution becomes then more focused if we have spatial knowledge. This knowledge can be expressed as:

1. ENST is west of the “Tolbiac station”;

2. ENST is not far from “Place d’Italie”.

Unfortunately the “Tolbiac station” is not represented on the map, but we know that it is between

"Place d'Italie" and "Porte d'Italie", i.e. to the South of the former and to the North of the later.

Coming back to our previous notations, ENST corresponds to point C, while "Tolbiac station" corresponds to point B and "Place d'Italie" to point A.

Each piece of knowledge can be expressed as a probability distribution. To model "not far from" we use a Gaussian distribution for C, centered on A (represented in Figure 14, top left). The distribution of B is chosen as a Gaussian distribution centered on the middle of the segment joining "Place d'Italie" and "Porte d'Italie" (Figure 14, top right).

The evaluation of $P(C, \alpha, \beta)$ as given in Equation 20 is represented in Figure 14, bottom left, with a maximum in the red area. The result is not very reliable and the center of the distribution is far from the actual position. Then we compute this probability by using the localization with respect to "Porte d'Italie" (Figure 14, bottom middle). The result may still appear not satisfactory. Therefore we use now both pieces of information. Their combination leads to a very focused distribution centered on ENST, as shown on the bottom right map of Figure 14.

Now if we use more prior information about the geographical localization of ENST such as:

3. ENST is south west of "Place d'Italie"

The prior distribution $P(C)$ can be chosen to best suit this new information. We have used the one presented in Figure 15 left to evaluate $P(C, \alpha, \beta)$ by using the localization with respect to "Porte d'Italie" (see Figure 15 middle). This result is more precise than the previous one (Figure 14 bottom middle).

Now if we combine the two distributions obtained by using the localization with respect to "Porte d'Italie" and "Place d'Italie", we obtain an even more focused and precise distribution (Figure 15 right). This improvement was expected because we have used more information about the spatial position of ENST.

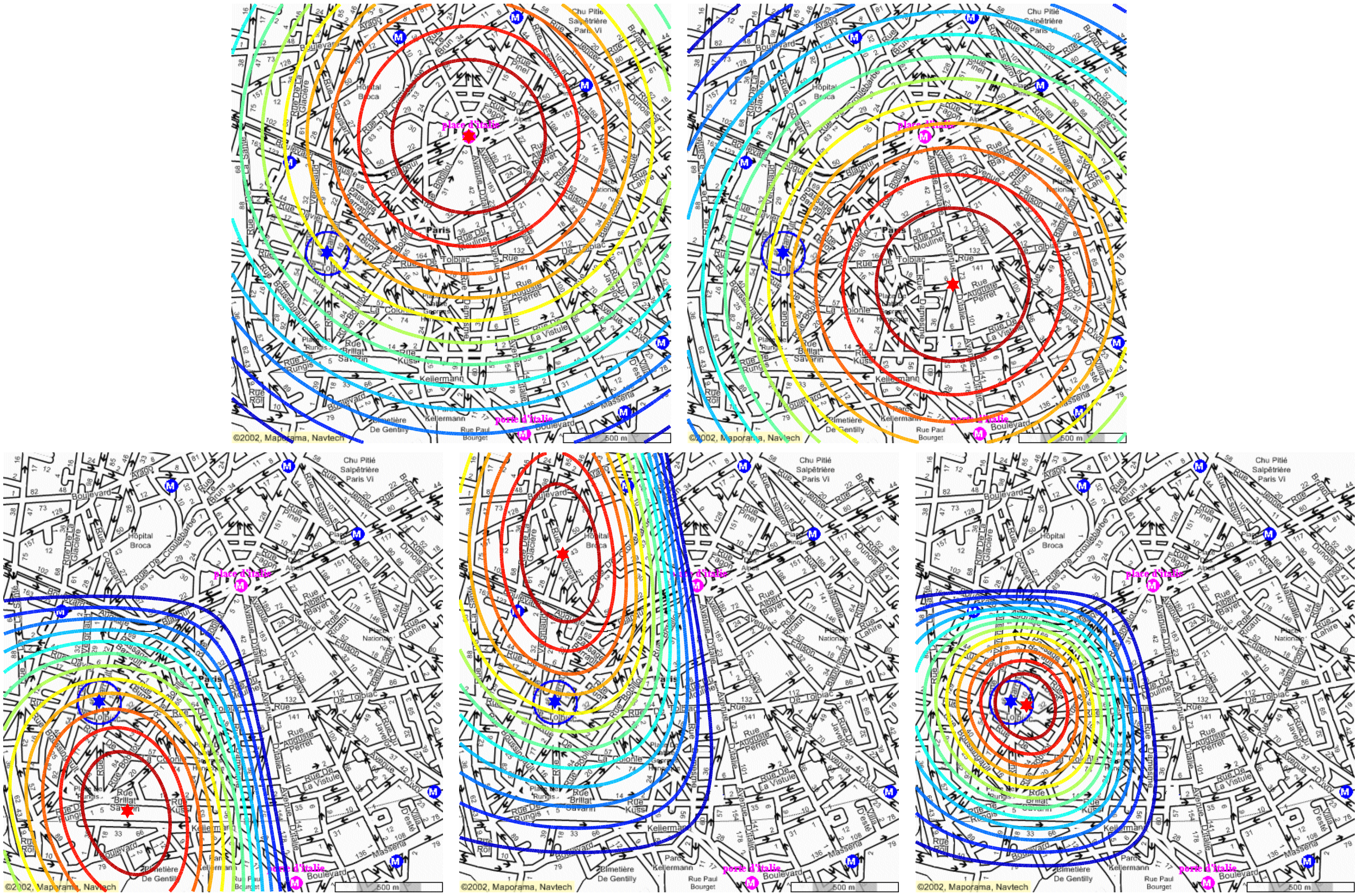


Fig. 14. Top left: distribution of point C (ENST) according to the information "B is not far from Place d'Ialie". Top right: distribution of point B (Tolbiac station) according to the information "C is between Place d'Italie and Porte d'Italie". Bottom (from left to right): localization from "Place d'Italie" and "West to Tolbiac station", localization from "Porte d'Italie", fusion of both distributions. The maximum of the distributions is represented by a red star.

VII. Conclusion

In the field of spatial reasoning and inference of spatial relations, we proposed in this paper a probabilistic formulation for this problem in the case of punctual objects. We derived analytical formulas in case of circular and square domains, and for different types of point distributions. The joint probability $P(C, \alpha, \beta)$ is valid for any distribution, while the conditional probability $P(\gamma|\alpha, \beta)$ has to be derived specifically for each distribution.

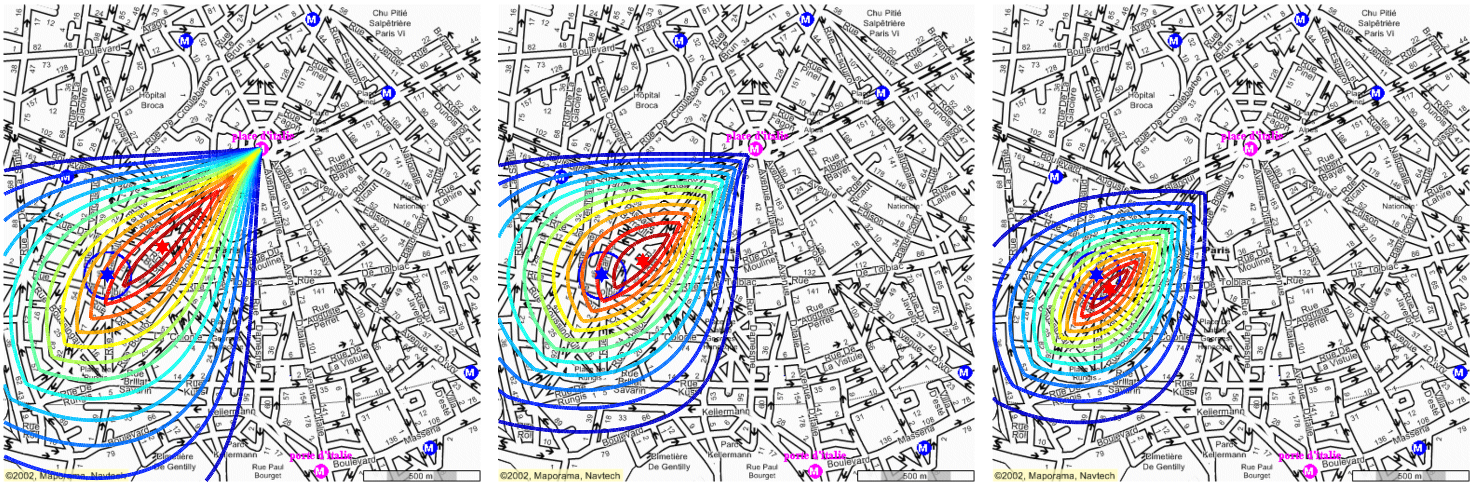


Fig. 15. Using directional information to define $P(C)$ (left). Localization from "Porte d'Italie" (middle). Combination of all pieces of knowledge (right).

In the general case, where we have to deal with spatial entities having any shape and spatial extension, two cases have to be distinguished. The first one corresponds to quite compact objects, either of small size, or far from each other. As a first approximation, such objects can be considered as points and the proposed approach applies directly. The second case corresponds to extended objects, for which the size is too large with respect to the distance between them to allow us to consider them as points. In such cases, the computation on points is no more sufficient. One possibility consists in combining the probability distribution with angle histograms [8]. This approach has been investigated in [10] but still needs to be further developed. In particular, estimating the conditional probability $P(\beta/\alpha)$ seems to be a necessary step.

Extensions to 3D are also possible, but the derivations are likely to be much more complex since positioning a point in a 3D space requires two angles instead of one. This is left for future work.

Foreseen applications concern for instance the localization of mobile phones as shown in our first simulated example. This problem is very close to the problem addressed in this paper and our hypotheses are consistent with this application. It typically belongs to the first case of extended objects.

Specific uncertainties linked to this particular problem should however be studied and, if it happens to be necessary, included in our model. This application could be useful typically in rescue issues.

Another possible application concerns recognition of objects from a structural model of the scene, where the model involves relationships between objects. This problem belongs in general to the second case of extended objects. Since it is very heavy to compute the relations between all objects, we can compute only a few and infer the others by the proposed approach (after a suitable extension to general objects, as mentioned above). Only those with a high probability with respect to the model can be kept. This will save computation time and also guide the recognition by limiting the possible matches between scene structures and model structures.

References

[1] L. Vieu, "Spatial Representation and Reasoning in Artificial Intelligence," in *Spatial and Temporal Reasoning*, pp. 5–41. Dordrecht, Kluwer, 1997.

[2] A.G. Cohn, "Qualitative Spatial Representations," in *IJCAI99 Workshop on Adaptive Spatial Representations of Dynamic Environments*, 1999, pp. 33–52.

[3] G. Ligozat, "Reasoning about Cardinal Directions," *Journal of Visual Languages and Computing*, vol. 9, pp. 23–44, 1998.

[4] James F. Allen, "Maintaining Knowledge about Temporal Intervals," *Communications of the ACM*, vol. 26, no. 11, pp. 832–843, 1983.

[5] S.K. Chang, Q.Y. Shi, and C.W. Yan, "Iconic Indexing by 2D Strings," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 9, no. 3, pp. 413–428, 1987.

[6] C. Freksa and K. Zimmermann, "On the Utilization of Spatial Structures for Cognitively Plausible and Efficient Reasoning," in *Proceedings of the IEEE International Conference on Systems, Man and Cybernetics*, Chicago, Oct. 1992, pp. 261–266.

[7] J. Freeman, "The Modeling of Spatial Relations," *Computer Graphics and Image Processing*, vol. 4, pp. 156–171, 1975.

[8] K. Miyajima and A. Ralescu, "Spatial Organization in 2D Segmented Images: Representation and Recognition of Primitive Spatial Relations," *Fuzzy Sets and Systems*, vol. 65, pp. 225–236, 1994.

[9] J. M. Keller and X. Wang, "Comparison of Spatial Relation Definitions in Computer Vision," in *ISUMA-NAFIPS'95*, College Park, MD, Sept. 1995, pp. 679–684.

[10] R. Dehak, *Inférence quantitative des relations spatiales directionnelles*, Ph.D. thesis, Ecole Nationale Supérieure des Télécommunications, Paris, France, 2002.

[11] P. Matsakis and L. Wendling, "A New Way to Represent the Relative Position between Areal Objects," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 21, no. 7, pp. 634–642, 1999.

[12] J. M. Keller and X. Wang, "Learning Spatial Relationships in Computer Vision," in *FUZZ-IEEE'96*, New Orleans, 1996, pp. 118–124.

[13] L.T. Koczy, "On the Description of Relative Position of Fuzzy Patterns," *Pattern Recognition Letters*, vol. 8, pp. 21–28, 1988.

[14] I. Bloch, "Fuzzy Relative Position between Objects in Image Processing: A Morphological Approach," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 21, no. 7, pp. 657–664, 1999.

[15] I. Bloch and A. Ralescu, "Directional Relative Position between Objects in Image Processing: A Comparison between Fuzzy Approaches," *Pattern Recognition*, vol. 36, pp. 1563–1582, 2003.

[16] S. Dutta, "Approximate Spatial Reasoning: Integrating Qualitative and Quantitative Constraints," *International Journal of Approximate Reasoning*, vol. 5, pp. 307–331, 1991.